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Coherence-based inequality for the discrimination of three-qubit GHZ and W class

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Abstract

Quantum coherence and entanglement originate from the superposition principle. We derive a rigorous relation between the l_1-norm of coherence and concurrence, in that we show that the former is always greater than the latter. This result highlights the hierarchical relationship between coherence and concurrence, and proves coherence to be a fundamental and ubiquitous resource. We derive an analogous form of monogamy inequality, which is based on the partial coherence of the reduced two-qubit and reduced single qubit of the particular class of three-qubit state. Moreover, we provide coherence-based inequality for the classification of GHZ class and W class of three-qubit states. Finally, we provide theoretical discussion for the possible implementation of the scheme in an experiment.

1 Introduction

Quantum coherence and entanglement are arguably the most significant phenomena appeared in quantum mechanics that mark the departure from classical mechanics. Entanglement has no classical analogue, but unlike this purely quantum mechanical phenomenon, coherence is a familiar event in optics. Although quantum theory of coherence forms the foundation of the study and manipulation of optical coherence phenomenon, there is a significant difference between the two, which has been studied and demonstrated through the multipoint correlation functions [1] and the phase space representation of quantum mechanics [2]. This works well to distinguish between the classical and quantum phenomena, but fails to quantify the amount of coherence

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present in a given system. To overcome this caveat, recently the resource theory of coherence was formulated by Buamgratz et al. [3]. It provides a quantum information theoretic framework to quantify and manipulate coherence levels in a system. The proper measure of coherence is needed to quantify the amount of coherence present in a quantum system. To probe this, they prescribed some postulates that an ideal measure of quantum coherence must satisfy. This has prompted various applications of quantum coherence to variegated fields such as thermodynamics [4], quantum metrology and sensing [5], one-way quantum computing [6] and quantum biology [7]. Quantum information protocols such as quantum secret sharing [8] and quantum private query [9] also exploit quantum coherence as a resource. Some of the important works to formulate an efficient resource theory of coherence are delineated in an extensive review [10].

Entanglement and coherence both arise from the superposition principle of quantum physics and are considered to be the key concepts for quantum technologies. Unlike entanglement, the amount of coherence depends on basis, and thus, the application of local unitary transformations on the quantum system may enhance the amount of coherence present in a system. In [11], a hierarchical relationship among quantum coherence, discord and entanglement is presented, which proves coherence as a fundamental manifestation of quantum correlations.

Superposition principle manifests itself in two ways in quantum mechanics: quantum coherence and quantum entanglement. Zhao et al. [12] have studied the relationship between coherence, concurrence and negativity for the particular class of two-qubit bipartite quantum states. The complementarity relation between the entanglement of formation and quantum coherence has been obtained by Pan et.al [13]. Stretslov et al. [14] have shown that there exist incoherent operations by which coherence can be converted to entanglement. A generalized process is given in [15], which shows general scheme to produce entanglement using nonclassicality as a resource.

The motivation of this work lies in the following facts: (i) Entanglement serves as a vital resource in various quantum information processing tasks such as teleportation and super-dense coding. But entanglement is quite expensive and difficult to prepare in comparison with other resources such as discord and coherence. Hence, it is imperative to determine a hierarchical relation between entanglement and coherence. (ii) The well-known monogamy inequality has been derived for quantum entanglement [16] and discord [17]. This motivates us to derive an analogous monogamy inequality based on the partial coherence of the two-qubit reduced state and single-qubit reduced state. (iii) There exist various methods based on entanglement by which GHZ class and W class can be distinguished, but there does not exist any method based on coherence by which we can distinguish GHZ class and W class. This is the driving force for the derivation of coherence-based inequality that may help to discriminate GHZ class and W class. The discrimination between GHZ class and W class is important because it is known that the state of the form $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ belongs to GHZ class always performed better in many quantum information processing tasks such as quantum teleportation than any three-qubit state that belongs to W-class. In [18], authors have studied the discrimination of three-qubit GHZ and W class of states.

This paper is organized as follows: In Sec. II, we have studied the relationship between coherence and concurrence of an arbitrary two-qubit state. In Sec. III, we have derived an inequality based on coherence that is analogous to concurrence-based monogamy inequality. Furthermore, we have constructed coherence-based inequality that may discriminate between GHZ class and W class. We conclude in Sec. IV.

2 Hierarchical relationship between coherence and concurrence of an arbitrary two-qubit state

In this section, we study the hierarchical relationship between the concurrence of an arbitrary two-qubit bipartite quantum state and its l_1-norm of coherence.
The l_1-norm of quantum coherence is defined as summation of modulus of the off-diagonal terms of given quantum state described by the two-qubit density matrix ρ [3],

$$C_{l_1}(\rho) = \sum_{i,j,i \neq j} |\rho_{ij}|, \quad i, j = 1, 2, 3, 4 \tag{1}$$

For any two-qubit density matrix ρ, concurrence can be defined as [19]

$$C(\rho) = max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\} \tag{2}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ denote the eigenvalues of the matrix $\rho\tilde{\rho}$, $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ referred to as the spin flipped density matrix and $\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ represent the Pauli matrix.

Theorem 1 *For any two-qubit entangled state described by the density matrix ρ, l_1-norm of quantum coherence $(C_{l_1}(\rho))$ is related to the concurrence of ρ $(C(\rho))$ as*

$$C^2(\rho) < 2(1 - S_L(\rho)) + C_{l_1}(\rho) \tag{3}$$

where $S_L(\rho) = \frac{4}{3}(1 - Tr(\rho^2))$ denote the linear entropy of ρ.

Proof Let us consider an entangled two-qubit quantum state described by the density operator ρ. It is given by

$$\rho = \frac{1}{4}[I \otimes I + \sum_{i=1}^{3} r_i\sigma_i \otimes I + I \otimes \sum_{i=1}^{3} s_i\sigma_i + \sum_{i,j=1}^{3} c_{ij}\sigma_i \otimes \sigma_j] \tag{4}$$

The two-qubit state ρ given in (4) can be written in the matrix form as

$$\rho = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{12}^* & t_{22} & t_{23} & t_{24} \\ t_{13}^* & t_{23}^* & t_{33} & t_{34} \\ t_{14}^* & t_{24}^* & t_{34}^* & t_{44} \end{pmatrix}, \sum_{i=1}^{4} t_{ii} = 1 \tag{5}$$

where $(*)$ denotes the complex conjugate.
The concurrence of the entangled state ρ is given by

$$\begin{aligned} C(\rho) &= \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \\ &\Rightarrow C(\rho) \leq \sqrt{\lambda_1} \\ &\Rightarrow C^2(\rho) \leq \lambda_1 \leq S_{max}(\rho\tilde{\rho}) \leq S_{max}(\rho) S_{max}(\tilde{\rho}) \end{aligned} \tag{6}$$

where $S_{max}(\rho\tilde{\rho})$ denotes the maximum singular value of the matrix $\rho\tilde{\rho}$. The last inequality follows from [20].
Inequality (6) can be further simplified by using the result $S_{max}(\rho) \leq \|\rho\|_2$, where $\|\rho\|_2^2 = Tr(\rho^2)$ [21]. Then, inequality (6) reduces to

$$C^2(\rho) \leq \|\rho\|_2 S_{max}(\tilde{\rho}) \leq \|\rho\|_2 \leq \sum_{i=1}^{4} t_{ii}^2 + C_{l_1}(\rho) \tag{7}$$

The proof of the last inequality is given in appendix. (For details, see Appendix-A.)
Since $t_{11} + t_{22} + t_{33} + t_{44} = 1$, we have

$$2 \sum_{i<j} t_{ii} t_{jj} = 1 - \sum_{i=1}^{4} t_{ii}^2 \tag{8}$$

The state parameter c_{33} in terms of t_{ii}, $(i = 1, 2, 3, 4)$ can be expressed as:

$$c_{33} = t_{11} - t_{12} - t_{33} + t_{44} \tag{9}$$

The linear entropy $S_L(\rho)$ of the state ρ is given by:

$$\begin{aligned} S_L(\rho) &= \frac{3}{4} - \frac{1}{4}[\sum_{i=1}^{3}(r_i^2 + s_i^2) + \sum_{i,j=1}^{3} c_{ij}^2] \\ &\leq \frac{3}{4} - \frac{1}{4} c_{33}^2 \\ &= \frac{3}{4} - \frac{1}{4}(t_{11} - t_{22} - t_{33} + t_{44})^2 \\ &= \frac{3}{4} - \frac{1}{4}\sum_{i=1}^{4} t_{ii}^2 - \frac{1}{2}(t_{11}t_{44} - t_{11}t_{22} \end{aligned}$$

$$
\begin{aligned}
&-t_{11}t_{33} + t_{22}t_{33} - t_{22}t_{44} - t_{33}t_{44}) \\
&< \frac{3}{4} - \frac{1}{4}\sum_{i=1}^{4} t_{ii}^2 + \frac{1}{2}\sum_{i<j} t_{ii}t_{jj}
\end{aligned} \tag{10}
$$

From (8) and (10), we get

$$
\sum_{i=1}^{4} t_{ii}^2 \leq 2(1 - S_L(\rho)) \tag{11}
$$

Using (11) in (7), we get

$$
C^2(\rho) < 2(1 - S_L(\rho)) + C_{l_1}(\rho) \tag{12}
$$

Hence, the theorem is proved. □

Corollary 1 *If ρ_A and ρ_B denote the reduced density operator of the composite state ρ_{AB} in $2 \otimes 2$-dimensional system, then [22]*

$$
C^2(\rho_{AB}) < 2(\frac{7}{4} - 2S_L(\rho_B) + \frac{1}{2}S_L(\rho_A)) + C_{l_1}(\rho_{AB}) \tag{13}
$$

$$
C^2(\rho_{AB}) < 2(\frac{7}{4} - 2S_L(\rho_A) + \frac{1}{2}S_L(\rho_B)) + C_{l_1}(\rho_{AB}) \tag{14}
$$

Proof If $S_L(\rho_A)$ and $S_L(\rho_B)$ denote the linear entropy of the reduced density operator ρ_A and ρ_B, then the following inequality holds [22]

$$
S_L(\rho_{AB}) \geq 2S_L(\rho_B) - \frac{1}{2}S_L(\rho_A) - \frac{3}{4} \tag{15}
$$

$$
S_L(\rho_{AB}) \geq 2S_L(\rho_A) - \frac{1}{2}S_L(\rho_B) - \frac{3}{4} \tag{16}
$$

Thus, Corollary-1 can be derived using results (15), (16) and Theorem-1 given in (3). □

3 Coherence-based inequality analogous to the concurrence-based inequality derived by Coffman et.al.

In this section, we will derive an inequality that provides us the upper bound of the sum of the coherences of the reduced two-qubit of the particular class of pure three-qubit system. This inequality is derived in the spirit of the seminal work by Coffman et.al. [16].

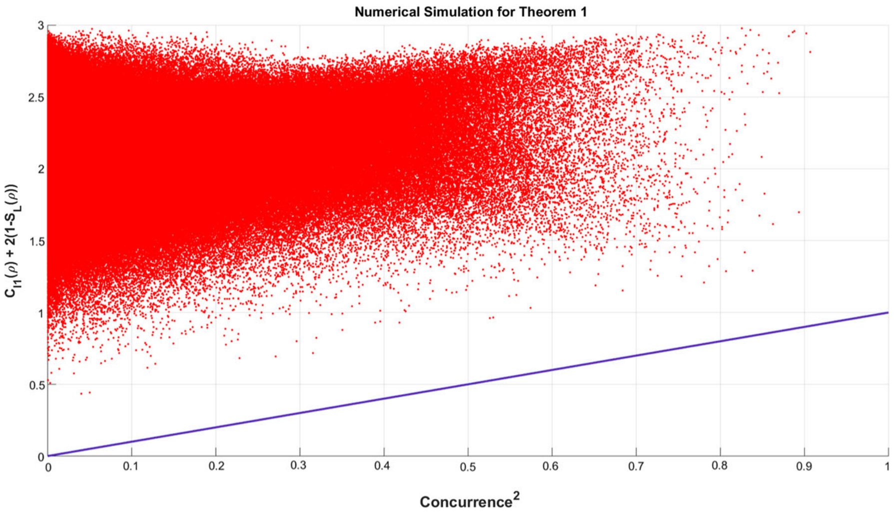

Fig. 1 The obtained results are backed up by the numerical analysis of 10^8 randomly generated density matrices. The blue line is $y = x$, and all the points lie in the region $y \geq x$, which corroborates Theorem 1

For a pure three-qubit state $|\psi\rangle_{ABC}$, Coffman et al. have derived an inequality based on the concurrence between the qubits A and B and the qubits A and C. Mathematically, this inequality can be expressed as [16]:

$$C_{AB}^2 + C_{AC}^2 \leq C_{A(BC)}^2 \tag{17}$$

where C_{AB} and C_{AC} denote the partial concurrences that measure the amount of entanglement in the reduced two-qubit mixed state described by the density operators ρ_{AB} and ρ_{AC}, respectively, of the pure three-qubit state $|\psi\rangle_{ABC}$. $C_{A(BC)}$ denotes the concurrence between subsystems A and BC. The inequality (17) is also known as monogamous inequality. Here, our aim is to construct the coherence-based inequality analogous to the concurrence-based inequality (17).

Any state vector in a three-qubit system can be spanned by eight computational basis vectors, and thus, we require seven parameters to characterize a general three-qubit quantum state. But Acin et al. [23] have deduced the canonical form of three-qubit state and shown that it can be represented by five parameters only. This idea was later on proved to be true for multipartite states by Cateret et al. [24].

The canonical form of three-qubit state can be expressed as [23]:

$$|\psi\rangle_{ABC}^{(\theta)} = \lambda_0 |000\rangle + \lambda_1 e^{i\theta} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle \tag{18}$$

where the state parameters $\lambda_i \geq 0$, $(i = 0, 1, 2, 3, 4)$ and the phase factor $0 \leq \theta \leq \pi$. The normalization condition gives

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1 \tag{19}$$

If there is no phase factor, i.e., $\theta = 0$, then the three-qubit state (18) reduces to a particular class, which is represented by:

$$
\begin{aligned}
|\psi\rangle^{(0)}_{ABC} = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle \\
+\lambda_4 |111\rangle
\end{aligned}
\tag{20}
$$

The relations between the state parameters and the partial concurrences of the pure three-qubit state $|\psi\rangle^{(0)}_{ABC}$ are invariants under local unitary transformation, and these invariant relations are given by [25]

$$
\begin{aligned}
C_{AB} &= 2\lambda_0\lambda_3 \\
C_{AC} &= 2\lambda_0\lambda_2
\end{aligned}
\tag{21}
$$

l_1 norm of coherence for the reduced two-qubit states and single-qubit state is described by the density operators ρ_{AB}, ρ_{AC} and ρ_A, respectively, of the pure three-qubit state $|\psi\rangle^{(0)}_{ABC}$ that are given by

$$C_{l_1}(\rho_{AB}) = 2(\lambda_0\lambda_1 + \lambda_0\lambda_3 + \lambda_1\lambda_3 + \lambda_2\lambda_4) \tag{22}$$

$$C_{l_1}(\rho_{AC}) = 2(\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2 + \lambda_3\lambda_4) \tag{23}$$

$$C_{l_1}(\rho_A) = 2\lambda_0\lambda_1 \tag{24}$$

Squaring (22) and (23) and then adding, we get

$$
\begin{aligned}
C^2_{l_1}(\rho_{AB}) + C^2_{l_1}(\rho_{AC}) &= 4[(\lambda_0(\lambda_1 + \lambda_3) + \lambda_1\lambda_3 + \lambda_2\lambda_4)^2 \\
&\quad +(\lambda_0(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 + \lambda_3\lambda_4)^2] \\
&= C^2_{AB} + C^2_{AC} + 2C^2_{l_1}(\rho_A) \\
&\quad +(\text{Sum of positive terms})
\end{aligned}
\tag{25}
$$

Equation (25) can also be expressed as:

$$
\begin{aligned}
C^2_{l_1}(\rho_{AB}) + C^2_{l_1}(\rho_{AC}) - 2C^2_{l_1}(\rho_A) &= \text{Sum of all finite positive} \\
&\quad\ \text{numbers} \\
&\geq 0
\end{aligned}
\tag{26}
$$

Hence, the required inequality is given by:

$$C^2_{l_1}(\rho_{AB}) + C^2_{l_1}(\rho_{AC}) \geq 2C^2_{l_1}(\rho_A) \tag{27}$$

Inequality (27) can be considered as an analogous form of the concurrence-based monogamous inequality (17). Inequality (27) holds for the particular type of large class of pure three-qubit state $|\psi\rangle^{(0)}_{ABC}$.

4 Few inequalities based on coherence

In this section, we will derive the coherence-based inequality that may be used to discriminate GHZ class and W class of pure three-qubit state. Then, we also characterize GHZ class of state based on few coherence-based inequalities.

4.1 Discrimination of pure three-qubit GHZ and W class using coherence-based inequality

GHZ class and W class represent two genuine entangled class of three-qubit pure state, which are inequivalent under stochastic local operation and classical communication (SLOCC). The amount of entanglement in three-qubit state that belongs to GHZ class can be quantified by the nonzero value of the three-tangle denoted by τ. For any pure three-qubit state $|\psi\rangle_{ABC}$, it can be defined as residual entanglement [16]

$$\tau = C^2_{A(BC)} - C^2_{AB} - C^2_{AC} \tag{28}$$

The tangle for the state $|\psi\rangle^{\theta}_{ABC}$ can be calculated as:

$$\tau_{|\psi\rangle^{\theta}_{ABC}} = 4\lambda_0^2\lambda_4^2 \tag{29}$$

If the state parameters λ_0 and λ_4 are nonzero, then the state that belongs to GHZ class can be expressed in the following form:

$$\begin{aligned} |\psi\rangle^{(0)}_{GHZ} = \lambda_0\,|000\rangle + \lambda_1\,|100\rangle + \lambda_2\,|101\rangle + \lambda_3\,|110\rangle \\ +\lambda_4\,|111\rangle \end{aligned} \tag{30}$$

The three-tangle vanishes for W class of states. Therefore, either $\lambda_0 = 0$ or $\lambda_4 = 0$ for W class of states. From (21), we can observe that if we take $\lambda_0 = 0$, then $C_{AB} = C_{AC} = 0$. Therefore, it would be advisable to take $\lambda_4 = 0$ for W class of state, and it is expressed in the form:

$$|\psi\rangle^{(0)}_{W} = \lambda_0\,|000\rangle + \lambda_1\,|100\rangle + \lambda_2\,|101\rangle + \lambda_3\,|110\rangle \tag{31}$$

Thus, it may appear that tangle can be a suitable candidate for the classification of GHZ class and W class. But since tangle remains zero for three-qubit biseparable and separable classes of states, it is not possible to conclude that the given class represents a W class if the tangle is zero. Thus, we derive here coherence-based inequality that may be used to classify pure three-qubit GHZ and W class of states.
Let us first recall (20), which represent the canonical form of pure three-qubit state $|\psi\rangle^{(0)}_{ABC}$. To start with the derivation of the inequality, let us consider the expression $C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_{AC})$, which is given by

$$C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_{AC}) = 2(\lambda_3 - \lambda_2)(\lambda_0 + \lambda_1 - \lambda_4) \tag{32}$$

Now we can consider two cases based on the sign of the expression $(\lambda_3 - \lambda_2)$.
Case-I: If $\lambda_3 - \lambda_2 \geq 0$, then we can observe the following points considered below:
(i) $C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_{AC}) \geq 0$, for every state belong to W class given by $|\psi\rangle_W^{(0)}$.
(ii) $C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_{AC}) < 0$, for at least one state belong to GHZ class given by $|\psi\rangle_{GHZ}^{(0)}$.
Case-II: If $\lambda_3 - \lambda_2 < 0$, then we have the following:
(i) $C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_{AC}) < 0$, for every state belong to W class given by $|\psi\rangle_W^{(0)}$.
(ii) $C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_{AC}) \geq 0$, for at least one state belong to GHZ class given by $|\psi\rangle_{GHZ}^{(0)}$.

4.2 Few results on the characterization of GHZ class

We discuss here few results, which will be applicable only for the states that belong to GHZ class.

Result-1: If the state belongs to GHZ class and choose the state parameters λ_0, λ_1 and λ_4 in such a way so that $\lambda_0 + \lambda_1 - \lambda_4 < 0$ holds, then

$$C_{AB}^2 + C_{AC}^2 < 2(2 - S_L(\rho_{AB}) - S_L(\rho_{AC}) + C_{l_1}(\rho_{AC})) \tag{33}$$

Proof Let us consider any state that belongs to $|\psi\rangle_{GHZ}^{(0)}$. Then, using the result in Theorem-1, the sum of the partial concurrences C_{AB} and C_{AC} can be expressed as:

$$\begin{aligned} C_{AB}^2 + C_{AC}^2 &\leq 2[2 - S_L(\rho_{AB}) - S_L(\rho_{AC})) + C_{l_1}(\rho_{AB}) \\ &\quad + C_{l_1}(\rho_{AC})] \\ &< 2[2 - S_L(\rho_{AB}) - S_L(\rho_{AC}) \\ &\quad + C_{l_1}(\rho_{AC})] \end{aligned} \tag{34}$$

If $\lambda_0 + \lambda_1 - \lambda_4 < 0$ holds for the state that belongs to GHZ class $|\psi\rangle_{GHZ}^{(0)}$, then we have $C_{l_1}(\rho_{AB}) < C_{l_1}(\rho_{AC})$, and we achieved the last inequality. Hence proved. □

Result-2: If the state belong to GHZ class and the inequality $\lambda_0 + \lambda_1 - \lambda_4 < 0$ holds for some state parameters λ_0, λ_1 and λ_4, then

$$C_{l_1}(\rho_A) < C_{l_1}(\rho_{AC}) \tag{35}$$

Proof Recalling (27) and re-expressing it as

$$\frac{C_{l_1}^2(\rho_{AB}) + C_{l_1}^2(\rho_{AC})}{2} \geq C_{l_1}^2(\rho_A) \tag{36}$$

Using AM-GM inequality on $C_{l_1}^2(\rho_{AB})$ and $C_{l_1}^2(\rho_{AC})$, we get

$$\frac{C_{l_1}^2(\rho_{AB}) + C_{l_1}^2(\rho_{AC})}{2} \geq C_{l_1}(\rho_{AB})C_{l_1}(\rho_{AC}) \tag{37}$$

From (36) and (37), it is not clear that whether $C_{l_1}(\rho_{AB})C_{l_1}(\rho_{AC}) - C_{l_1}^2(\rho_A) \geq 0$ or $C_{l_1}(\rho_{AB})C_{l_1}(\rho_{AC}) - C_{l_1}^2(\rho_A) < 0$ holds. To investigate this, let us express the value of the expression $C_{l_1}(\rho_{AB})C_{l_1}(\rho_{AC}) - C_{l_1}^2(\rho_A)$ in terms of the state parameters. We have

$$\begin{aligned} C_{l_1}(\rho_{AB})C_{l_1}(\rho_{AC}) - C_{l_1}^2(\rho_A) &= 4\lambda_0\lambda_1\lambda_2(\lambda_0+\lambda_1) \\ &\quad +4\lambda_3(\lambda_0+\lambda_1)(\lambda_0\lambda_1 + \\ &\quad \lambda_0\lambda_2+\lambda_1\lambda_2) \end{aligned} \tag{38}$$

Since all $\lambda_i \geq 0$, we get

$$C_{l_1}(\rho_{AB})C_{l_1}(\rho_{AC}) \geq C_{l_1}^2(\rho_A) \tag{39}$$

If $\lambda_0 + \lambda_1 - \lambda_4 < 0$ holds for the state that belongs to GHZ class $|\psi\rangle_{GHZ}^{(0)}$, then we have $C_{l_1}(\rho_{AB}) < C_{l_1}(\rho_{AC})$, and using the result given in (39), we get

$$\begin{aligned} &C_{l_1}^2(\rho_A) \leq C_{l_1}^2(\rho_{AC}) \\ &\Rightarrow C_{l_1}(\rho_A) < C_{l_1}(\rho_{AC}) \end{aligned} \tag{40}$$

Hence proved. □

4.3 Experimental realization of the inequality $\lambda_0 + \lambda_1 - \lambda_4 < 0$

In the previous sections, we have seen that the inequality $\lambda_0 + \lambda_1 - \lambda_4 < 0$ plays an important role in the discrimination of GHZ class and W class and also take part in the characterization of GHZ class. By seeing its importance in the characterization and classification problem, we provide here the theoretical prescription of the experimental realization of the inequality $\lambda_0 + \lambda_1 - \lambda_4 < 0$.
Multiplying by $\lambda_0 > 0$ both sides of the inequality $\lambda_0 + \lambda_1 - \lambda_4 < 0$, we get

$$\begin{aligned} &\lambda_0^2 + \lambda_0\lambda_1 - \lambda_0\lambda_4 < 0 \\ &\quad \Rightarrow \sqrt{\tau} \geq 2\lambda_0(\lambda_0+\lambda_1) \\ &\quad \Rightarrow \langle O\rangle_{|\psi\rangle_{ABC}^{(0)}} > \langle O_1\rangle_{|\psi\rangle_{ABC}^{(0)}} + \langle O_2\rangle_{|\psi\rangle_{ABC}^{(0)}} \end{aligned} \tag{41}$$

where the operators O, O_1 and O_2 can be decomposed in terms of Pauli matrices as

$$O = 2(\sigma_x \otimes \sigma_x \otimes \sigma_x) \tag{42}$$

$$O_1 = 2(\sigma_x \otimes \sigma_z \otimes \sigma_z) \tag{43}$$

$$O_2 = \frac{1}{4}[(I+\sigma_z)\otimes(I+\sigma_z)\otimes(I+\sigma_z)] \tag{44}$$

The expectation value of the operators $\langle O\rangle_{|\psi\rangle_{ABC}^{(0)}}$, $\langle O_1\rangle_{|\psi\rangle_{ABC}^{(0)}}$, $\langle O_2\rangle_{|\psi\rangle_{ABC}^{(0)}}$ paves a way for the possible implementation of the technique to distinguish GHZ class and W class. A classification protocol introduced in [26] has been implemented on an NMR-based quantum information processor by Singh et al. [27]. Thus, we believe that the classification scheme studied in this work may be implemented in NMR-based experiment

5 Conclusion

To summarize, we have illustrated a rigorous proof that l_1-norm of coherence is greater than concurrence for a general two-qubit system. In the context of partial concurrence-based monogamy inequality, we have designed an analogous form of monogamy inequality based on partial coherence of the special large class of three-qubit pure state. We have also derived the partial coherence-based inequality to distinguish between GHZ and W class of states, and further we have characterized three-qubit GHZ class on the basis of the constructed inequality. We have corroborated our theoretical efforts by providing an experimental scheme to implement our proposal. We believe that this work may deepen our understanding of coherence as a resource and may provide us with better insights to manifest quantum technologies.

Data Availability Statement Data sharing was not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix-A

To prove $Tr(\rho^2) \leq \sum_{i=1}^{4} t_{ii}^2 + C_{l_1}(\rho)$

In this section, we will provide the proof of the inequality $Tr(\rho^2) \leq \sum_{i=1}^{4} t_{ii}^2 + C_{l_1}(\rho)$. To achieve our goal, we recall an arbitrary two-qubit quantum state described by the density operator ρ given in (5).
l_1 norm of coherence of ρ is given by

$$C_{l_1}(\rho) = 2 \sum_{i,j=1,i\neq j}^{4} |t_{ij}| \tag{45}$$

$Tr(\rho^2)$ is given by

$$\begin{aligned} Tr(\rho^2) &= \sum_{i=1}^{4} t_{ii}^2 + 2(|t_{12}|^2 + |t_{13}|^2 + |t_{14}|^2 + |t_{23}|^2 \\ &\quad + |t_{24}|^2 + |t_{34}|^2) \\ &\leq \sum_{i=1}^{4} t_{ii}^2 + 2(|t_{12}| + |t_{13}| + |t_{14}| + |t_{23}| \\ &\quad + |t_{24}| + |t_{34}|) \\ &= \sum_{i=1}^{4} t_{ii}^2 + C_{l_1}(\rho) \end{aligned} \tag{46}$$

Thus, we have

$$Tr(\rho^2) \leq \sum_{i=1}^{4} t_{ii}^2 + C_{l_1}(\rho) \tag{47}$$

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