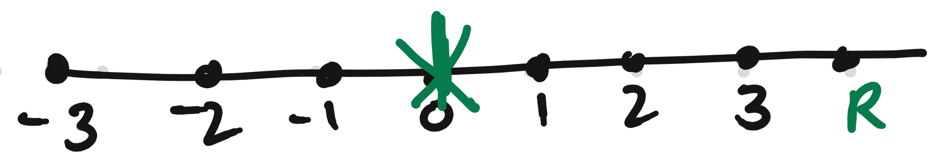
Lieb-Robinson Bound

$$H = \sum_{i \in L} \left[J_i Z_i Z_{i+1} + h_i X_i + g Z_i + K_i Y_i Y_{i+1} \right]$$



over long timescales H has an emergent notion of locality

If we perturb $|0\rangle$ at $t=0$, will I detect it at $|R\rangle$ at later time t

↳ $\langle [Z_R, X_0(t)] \rangle \neq 0$? how sensitive is correlation to earlier.

↳ approx hold if $R < vt$ ↳ **LB** velocity

→ Why should these bounds exist?

(1) choose $J, K, g \dots$ s.t. we have non-interacting system where

$\varepsilon(K) = 2h(1 - \cos K)$ → how fast can wavepacket spread?

↓ $\max(v_g) \to 2h = \text{const}$.

emergent speed limit on spread of info

(2) $H = \sum_{i \in L} h(|i\rangle\langle i+1| + h.c.)$ & $|\psi(0)\rangle = |0\rangle$

At what $t = T$ is $P(x \geq R) \geq \frac{1}{2} \Rightarrow \sum_{|x| > R} |\langle x|\psi(T)\rangle|^2$

$$\langle x| e^{-iHt} |0\rangle = \langle x| \sum_{n=0}^{\infty} \frac{(-iHt)^n}{n!} |0\rangle$$

To reach R we need atleast first R terms in perturbation series

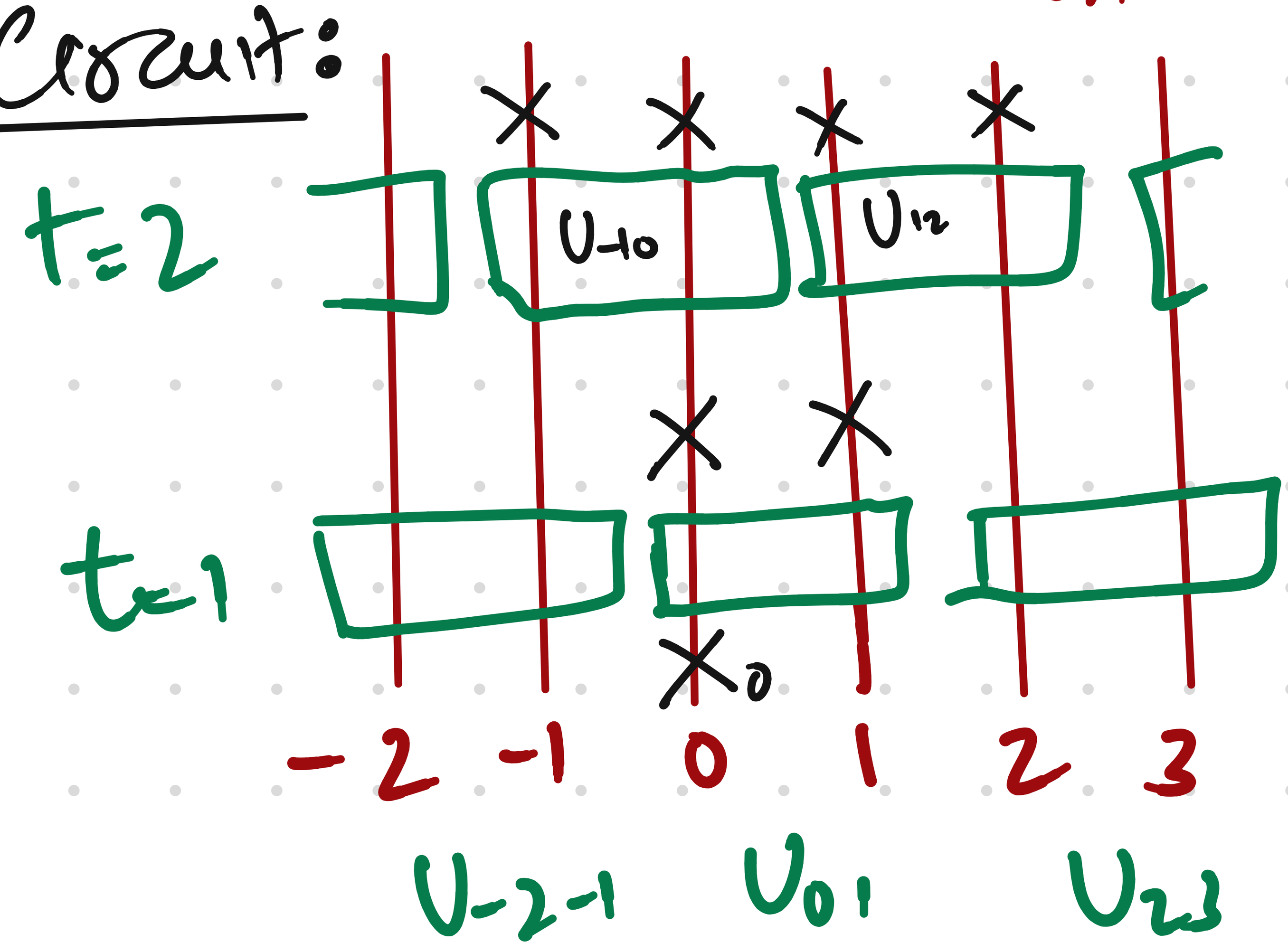
short t: $\langle R | \psi(T) \rangle \simeq \frac{(-iht)^R}{R!} + \mathcal{O}(t^{R+1})$

$|\quad " \quad| \sim \left(\frac{eht}{R}\right)^R \qquad R! > \frac{R^R}{e^R}$ (Stirling)

should be 1 → $ehT \geq R$

"V_{LR}" $= eh$

(3) Many Body Circuit:



$|\psi(0)\rangle = |0 \dots x_0 x_1 x_2 \dots\rangle$

$|\psi(t)\rangle = U_{-2-1} U_{01} U_{23} \dots |\psi(0)\rangle$

↳ looks nonlocally different

How do we think of locality in this system?

→ Apply X at $t=0$ and then at what sites this operation changes the functi.

$X_0(1) = X_0 X_1$
$X_0(2) \sim \sum X_{-1} X_0 X_1 X_2$ } after t time steps it acts on $\leq 2t$ sites

distance $\leq t$ from $x=0$; $V_{LR} = 1$

Perturbation A_0 at $t=0$: can only influence site 'R' when correlation is $\neq 0$

${}_{pert}\langle \psi(t) | B_r | \psi(t) \rangle_{pert}$: $|\psi(t)\rangle_{pert} = U(t) . e^{iA_0 t} |\psi_0\rangle$

$U^{+} B_r U = B_r(t)$

$g \otimes g \ldots\ldots X_{r-t} \otimes \ldots X_{r+t} \otimes g \ldots g$

Taylor exp. → $\langle \psi_0 | B_r(t) | \psi_0 \rangle - i\varepsilon \langle \psi_0 | [A_0, B_r(t)] | \psi_0 \rangle + O(\varepsilon^2)$

Need $r \leq t$ to not vanish.

time it takes for a typical string (in the sum X) to reach r → butterfly velocity

↓

$Tr(-[A_0, B_r(t)]^2) = \sum_S |C_S|^2$

↓

S is non trivial at origin & anti commuting with A_0

$$\boxed{\text{Proof L-R bound}}$$

Goal: for any $|\psi_{1,2}\rangle$, $\wedge |\langle \psi_1 | [A_0, B(r(t)] | \psi_2 \rangle|$ should be small → factorial decay

$\sup_{\psi_{1,2}} = \| [A_0, B_r(t)] \| = \lambda_{max}$

Try #1: $B_r(t) = B_r + i[H, B_r]t + \ldots\ldots + \frac{(-it)^n}{n!}[H, \ldots [H, B_r]]$

$e^{iHt} . B_r e^{-iHt} =$

$[H_{r-1} + H_r, B_r]$

↓ nested commutators $O(n!)$ terms can't sum it

Try #2: $\| [A_0, B_r(t)] \|$

$= \| [A_0, U_\varepsilon^{+} B_r(t-\varepsilon) U_\varepsilon] \|$

$= \| [U_\varepsilon A_0 U_\varepsilon^{+}, B_r(t-\varepsilon)] \| = \| [A_0(-\varepsilon), B_r(t-\varepsilon)] \|$

$\simeq A_0 - i\varepsilon \underbrace{[H, A_0]}_{A_{-1,0} + A_{0,1}}$ → recursive relation can be obtained

$$C_{RS}(t) = \| [A_R, B_S(t)] \| \rightarrow \frac{d}{dt} C_{RS} \leq \sum h_{RR'} C_{R'S}$$

End: $C_{\{0\}\{R\}} \leq \left(\frac{2eht}{R} \right)^R$

Applications

- can't prepare interesting long range entanglement outside LR light cone
- Goal: product state $\rightarrow$ Bell or

$$e^{-iHt} |\ldots 00 \ldots\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \otimes |\psi\rangle$$

or

$$e^{-iHt} (\alpha|0\rangle + \beta|1\rangle) \otimes |\ldots\rangle = \left[\alpha \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) + \beta \frac{(|01\rangle + |10\rangle)}{\sqrt{2}} \right] \otimes |\psi\rangle$$

At time t: logical $X(t)(\alpha \leftrightarrow \beta)$ with X_R.

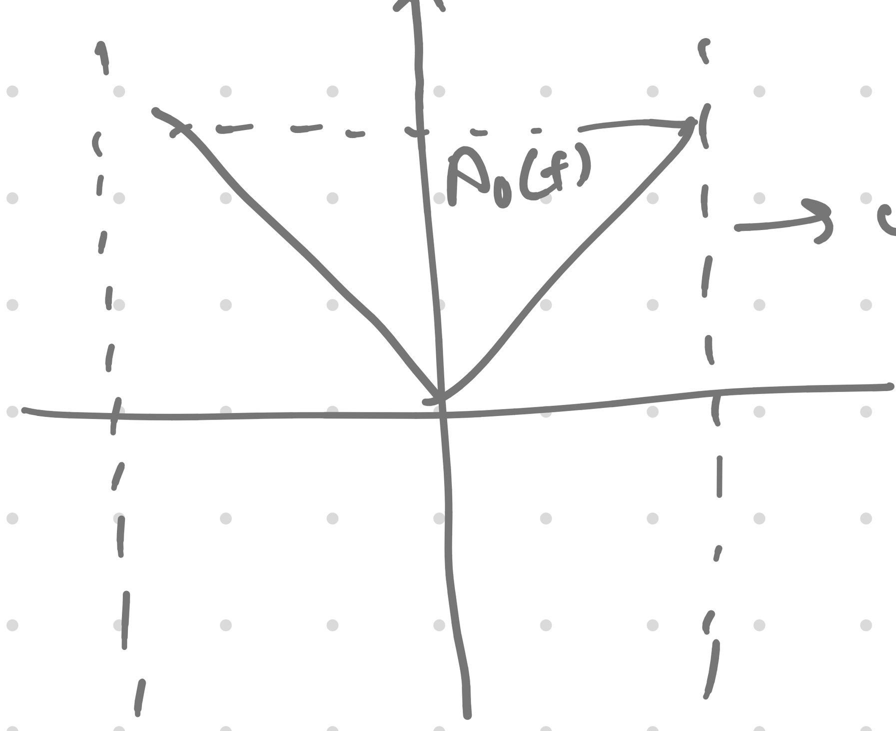
Swap order of operators with $Z(0) = Z_0$.

if we can do it then: $\| [X_R(t), Z_0] \| = 2$ $\quad$ $\because XZ = -ZX$

this means $R \leq V_{LR}\, t$

* Simulating Quantum Dynamics:

Let's say we calculate $\langle [A_0(t), B_r] \rangle$ on classical simulator upto an error ε. then

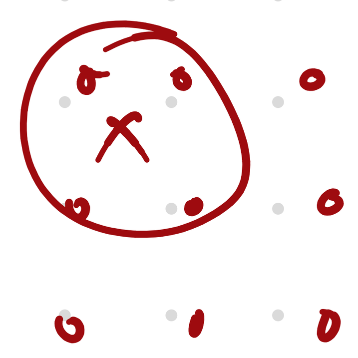


$\rightarrow$ we can truncate computational space to dotted lines

** Quantum simulation: HHKL Algo : uses LB to convert continuous time evolutions into discrete circuits

* H_0 gapped $\rightarrow$ finite correlation length in ground state

Going deeper into LB-Bounds

$H = \sum h_x$  $\text{diam}(X) = R$: $R=1$ nearest neighbours.

$O(t) = e^{iHt} O e^{-iHt} \rightarrow$ supported on X

$O(t)$ well approximated if $R \leq v_{LR} \cdot t$ on X

- for local gapped non-degenerate GS $\langle \Psi_{GS} | \quad | \Psi_{GS} \rangle = \langle \quad \rangle$

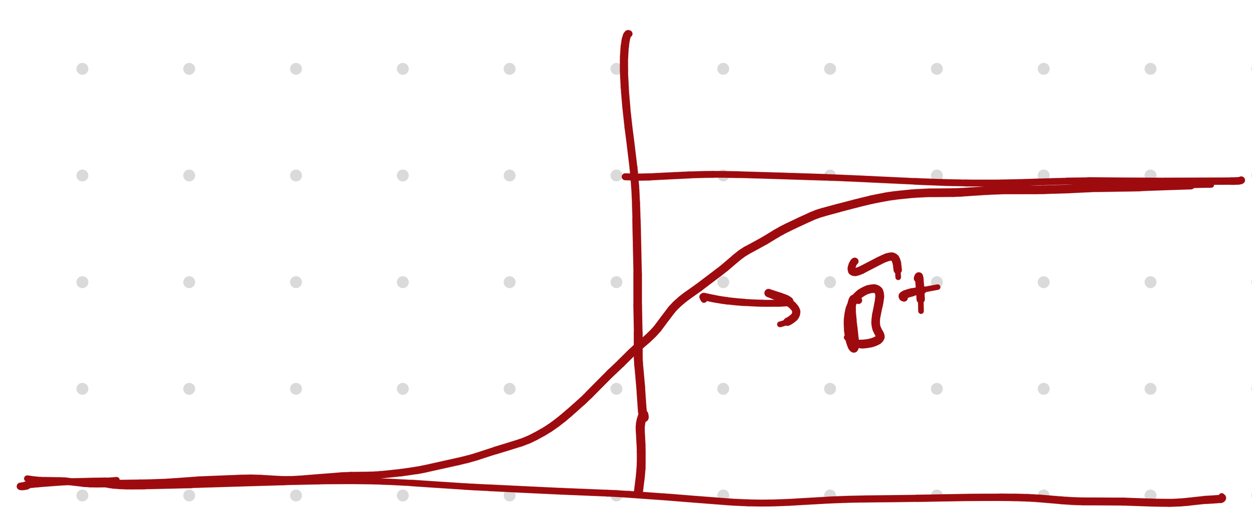
$$|\langle A_x B_y \rangle - \langle A_x \rangle \langle B_y \rangle| \leq e^{\left(\frac{dist(X,Y) \cdot \Delta}{2 v_{LR}}\right)}$$

B^+: positive energy part of B ; $(B^+) = B_{ij}$ if $E_i > E_j$ (raises energy)
$\qquad 0$ if $E_i < E_j$

$\langle AB \rangle = \langle AB^+ \rangle = \langle [AB^+] \rangle$

$B^+ = \frac{1}{2\pi i} \int dt \frac{B(t)}{t + i\varepsilon} \rightarrow$ picks +ve frequency

$\tilde{B}^+ = \quad " \qquad " \cdot e^{-t^2 \Delta} \rightarrow$ cutoff gaussian time.

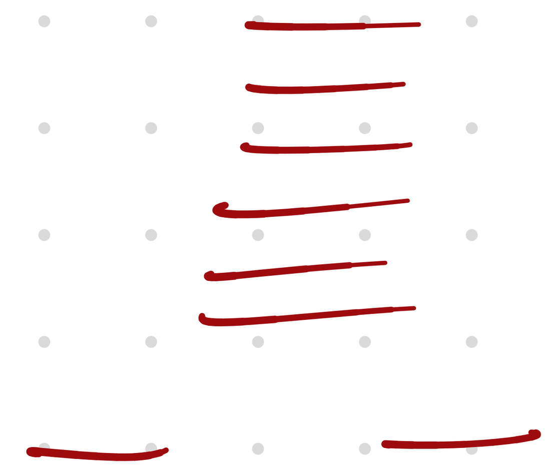


- make gaussian wide enough that $|B^+ - \tilde{B}^+|$ small when $E \sim \Delta$ then $\tilde{B}^+$ approximates B^+ quite well in this region. Integral is cutoff at large 't'. focus on small time evolution
- Use LB bounds to say within short distance this B commutes with A
- $\Delta \rightarrow$ fourier analysis $\rightarrow$ delocalized in time can be well approximated in short time $\rightarrow$ LB short range in space

Eg 2

$$H = J_1 \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \sum_i \vec{S}_i \cdot \vec{S}_{i+2}$$

Exactly solvable: $J_2 = \frac{J_1}{2}$



$L \to \infty$ $\leftarrow \Delta E$ [$\rightarrow$ 2 gen GS

ΔE is stable for weak perturbation in J_1, J_2

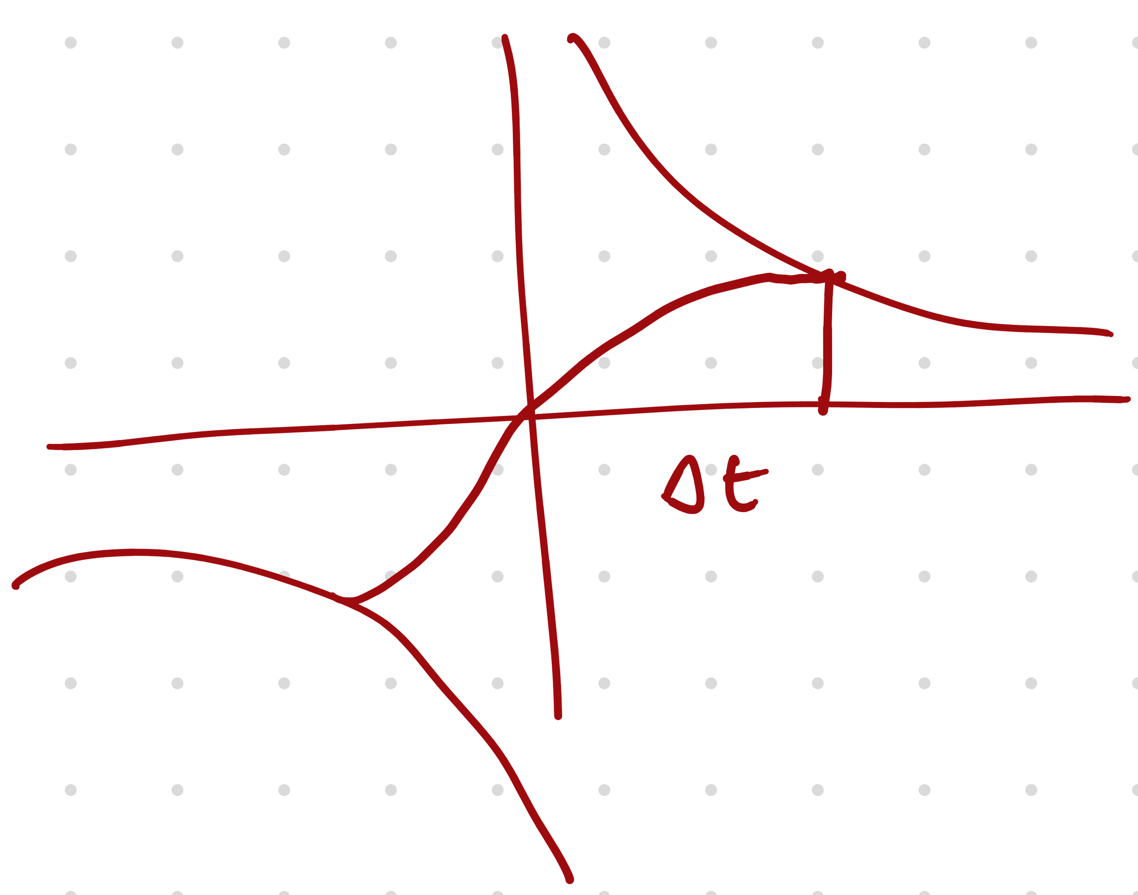
① Correlation decay: $[A_x(t), B_y]$ $\mapsto$ expect to be zero if X & Y are far apart

Proof: $$\langle [A_x(t), B] \rangle = \sum_{n>0} \langle \psi_0 | A_x | \psi_n \rangle \langle \psi_0 | B_y | \psi_0 \rangle e^{-iE_n t}$$

$E_0 = 0$ $$- \sum_{n>0} \langle \psi_0 | B_y | \psi_n \rangle \langle \psi_n | A_x | \psi_0 \rangle e^{iE_n t}$$

Quasi-Adiabatic Continuation

$$\partial_s \Psi_0 = \sum_{n>0} \frac{1}{E_0 - E_n} | \psi_n \rangle \langle \psi_n | \partial_s H_s | \psi_0 \rangle$$



$$\partial_s \Psi_0 = i \int dt . (\partial_s H_s)(t) f(t) . \Psi_0$$